

TALK #2: CATEGORY \mathcal{O}

1. SETUP

Throughout this paper G is a reductive Lie group over \mathbb{C} with Lie algebra \mathfrak{g} .

The main object of our studies is a Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$, which as a vector space is isomorphic to $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g}((t)) \oplus \mathbb{C} \cdot 1$, while the Lie bracket is given by $[fc, f'c'] = ff'[c, c'] + \text{Res}_{t=0}(f'df)\kappa(c, c') \cdot 1$ with $c, c' \in \mathfrak{g}$, $f, f' \in \mathbb{C}((t))$. We will also need its *polynomial* subalgebra $\widetilde{\mathfrak{g}}_\kappa$ which as a vector space is $\widetilde{\mathfrak{g}}_\kappa = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot 1$. Finally, we will also use subalgebra $\widehat{\mathfrak{g}}_\kappa^+ \subset \widehat{\mathfrak{g}}_\kappa$ defined by $\widehat{\mathfrak{g}}_\kappa^+ = \mathfrak{g}[[t]] \oplus \mathbb{C} \cdot 1$. Note that $\widehat{\mathfrak{g}}_\kappa^+$ is a trivial central extension of $\mathfrak{g}[[t]]$.

In what follows Λ is a lattice of integral weights of \mathfrak{g} , $\Lambda^+ \subset \Lambda$ is a subset of dominant integral weights, κ is a non-degenerate invariant bilinear form on \mathfrak{g} . For each $\lambda \in \Lambda$, V_λ will denote the irreducible \mathfrak{g} -module of highest weight λ . Finally we will need the critical level form on \mathfrak{g}_i equal to $\kappa_{crit, i} = -1/2 \cdot \kappa_{Kill, i}$, with $\kappa_{Kill, i}$ being a Killing form on \mathfrak{g}_i .

Remark 1. *This slightly differs from Kazhdan's talk and [3], where \mathfrak{g} was assumed to be simple. However, writing down any reductive Lie algebra \mathfrak{g} as $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{i=1, \dots, l} \mathfrak{g}_i$ with abelian \mathfrak{g}_0 and each other \mathfrak{g}_i being a simple Lie algebra we, in fact, have a decomposition of κ as well, i.e. we can write $\kappa = \bigoplus \kappa_i$ (as $\kappa|_{\mathfrak{g}_i \times \mathfrak{g}_j} = 0$ for $i \neq j$).*

Definition 1. *Representation ρ of $\widehat{\mathfrak{g}}_\kappa$ is called of level κ if $\rho(1) = Id$.*

Let us recall the basic definition of a *smooth representation*:

Definition 2. *A level κ representation $\rho : \widehat{\mathfrak{g}}_\kappa \rightarrow \text{End}(V)$ is called smooth if for any $v \in V$ there is $N = N(v)$, s.t. $t^n \mathfrak{g} \cdot v = 0$ for any $n > N$. We denote the category of smooth $\widehat{\mathfrak{g}}_\kappa$ -modules by $\widehat{\mathfrak{g}}_\kappa - \text{mod}$. For any representation V we define its smooth part by V_{sm} .*

We assume the notion of the scheme $G[[t]]$ is familiar to a reader (e.g. see [2], section 2.4).

We define the category KL_κ (of Kazhdan-Lusztig modules) as following:

Definition 3. *KL_κ is the category of modules for a Harish-Chandra pair $(G[[t]], \widehat{\mathfrak{g}}_\kappa - \text{mod})$.*

Unwinding this definition we see that KL_κ is a full abelian subcategory of $\widehat{\mathfrak{g}}_\kappa - \text{mod}$, s.t. $V \in \text{KL}_\kappa$ iff $t\mathfrak{g}[[t]]$ acts locally nilpotently and the action of \mathfrak{g} integrates to G .

Following notations from [3] and Kazhdan's talk we define:

Definition 4. *i) A linear span of $g_1 \dots g_N$ with $g_i \in t\mathfrak{g}[t]$ is denoted by $\mathcal{Q}_N \subset U(\widetilde{\mathfrak{g}}_\kappa) \subset U(\widehat{\mathfrak{g}}_\kappa)$;
ii) For any $\widetilde{\mathfrak{g}}_\kappa$ -module/ $\widehat{\mathfrak{g}}_\kappa$ -module V we define $V(N) := \{v \in V | \mathcal{Q}_N \cdot v = 0\}$;
iii) For any $\widetilde{\mathfrak{g}}_\kappa$ -module/ $\widehat{\mathfrak{g}}_\kappa$ -module V we define $V_{int} := \bigcup_N V(N)$;
iv) A $\widetilde{\mathfrak{g}}_\kappa$ -module/ $\widehat{\mathfrak{g}}_\kappa$ -module V is called integrable if $V = V_{int}$.*

Remark 2. *a) Our definition of \mathcal{Q}_N (and hence $V(N)$) slightly differs from the one adopted in [3], where it is defined as a span of $g_1 \dots g_N$ with $g_i \in t\mathfrak{g}$. However, in the case of a semi-simple \mathfrak{g} it is easy to see that $V(N)$ remain the same in both cases.*

b) While for a semi-simple \mathfrak{g} any integrable module is smooth, it's false for a reductive \mathfrak{g} .

Assumption 1. *In what follows we always assume $\kappa - \kappa_{crit}$ is non-degenerate.*

By this we mean that $\kappa_i - \kappa_{crit,i}$ are non-degenerate bilinear forms on \mathfrak{g}_i ($0 \leq i \leq l$).

The importance of this assumption is motivated by the *Sugawara Construction*. It has been mentioned last time and will be elaborated more in Andrei's talk. The only thing we will need today is the existence of the pairwise commuting operators $\{L_0^{(i)}\}_{0 \leq i \leq l}$ satisfying the property $[L_0^{(i)}, t^k c] = kt^k c$ for any $k \in \mathbb{Z}, c \in \mathfrak{g}_i$ and $[L_0^{(i)}, t^k \mathfrak{g}_j] = 0$ ($i \neq j$).¹ Each operator $L_0^{(i)}$ is given by the formula $L_0^{(i)} = \sum_{j>0} \sum_p (t^{-j} c_p^{(i)})(t^j c_p^{(i)}) + \sum_p c_p^{(i)} c_p^{(i)}$ with $c_p^{(i)}$ being an orthonormal basis of \mathfrak{g}_i with respect to the bilinear form $-2(\kappa_i - \kappa_{crit,i})$ (it is easy to see that it doesn't depend on the choice of the basis $c_p^{(i)}$).² Though it is an element of the completion of $U(\widehat{\mathfrak{g}}_\kappa)$ it's action on any smooth $\widehat{\mathfrak{g}}_\kappa$ -representation is well-defined.

Remark 3. a) Note that $L_0 := \sum_{0 \leq i \leq l} L_0^{(i)}$ satisfies relations $[L_0, t^k c] = kt^k c, k \in \mathbb{Z}, c \in \mathfrak{g}$ as in the case of a semi-simple \mathfrak{g} . We adopt this notation in what follows.

b) Peculiarity of an abelian component \mathfrak{g}_0 is that we have an inclusion $\mathfrak{g}_0 \subset \mathfrak{z}(\widehat{\mathfrak{g}})$. In what follows this fact plays the same role as operators $L_0^{(i)}$ for $\mathfrak{g}_i, i \neq 0$, while $L_0^{(0)}$ turns out to be pretty useless.

Definition 5. i) Given a finite dimensional $G[[t]]$ -module M we define a *generalized Weyl module* $M^\kappa := \text{Ind}_{\widehat{\mathfrak{g}}_\kappa^+}^{\widehat{\mathfrak{g}}_\kappa}(M)$;

ii) Considering a finite dimensional irreducible G -representation V_λ ($\lambda \in \Lambda^+$) as a $G[[t]]$ -module we call V_λ^κ *Weyl module*.

2. KEY PROPERTIES OF GENERALIZED WEYL MODULES

For the rest of the talk we make a crucial assumption:

Assumption 2. In what follows we always assume $\kappa - \kappa_{crit}$ **is not a positive rational**, meaning that κ_0 is non-degenerate and $c_i + 1/2 \notin \mathbb{Q}_{\geq 0}$, with c_i defined via $\kappa_i = c_i \cdot \kappa_{Kill,i}$.

The following Proposition from a previous talk summarizes basic properties of generalized Weyl modules.

Proposition 1. a) Any generalized Weyl module has a finite filtration with consequent quotients being Weyl modules;

b) Any generalized Weyl module has finite length;

c) Any Weyl module V_λ^κ has a unique irreducible quotient L_λ^κ and $L_\lambda^\kappa \not\cong L_\mu^\kappa$ for $\lambda \neq \mu$;

d) For a generalized Weyl module V the action of Sugawara operators $L_0^{(i)}$ induces a decomposition into a countable direct sum of finite dimensional generalized eigenspaces $V = \bigoplus_{\chi \in \mathbb{C}^{l+1}} \chi V$.

Remark 4. As can be observed during the proof, parts a),c),d) remain valid for any non-critical level κ , while it's only b) that requires our assumption on $\kappa - \kappa_{crit}$ to be non-positive rational.

Once our results are stated in the full generality for the case of reductive \mathfrak{g} , in the rest of this talk we will assume \mathfrak{g} is simple in order to make our arguments shorter and more understandable. In particular, there is only one Sugawara operator L_0 .

¹In other words we pick a Sugawara operator L_0 for each component \mathfrak{g}_i .

²To pick such a basis we essentially need non-degeneracy of $\kappa_i - \kappa_{crit,i}$. That's why Sugawara Construction fails at the *critical level*.

Proof. Claim a) is trivial. Indeed, it is easy to see that any finite-dimensional $G[[t]]$ -module M possesses a filtration by $\widehat{\mathfrak{g}}_\kappa^+$ -modules $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$, s.t. $t\mathfrak{g}[[t]]$ acts trivially on every quotient M_i/M_{i+1} . Hence the representation of $\mathfrak{g}[[t]]$ in M_i/M_{i+1} factors through its quotient \mathfrak{g} via evaluation map. Since any finite dimensional G -representation decomposes into the sum of G -irreducible modules we can refine out filtration $\{M_i\}$ in such a way that M_i/M_{i+1} is irreducible as a G -module. This filtration is obviously finite since M is finite dimensional. Thus we get the desired filtration $M^\kappa = M_0^\kappa \supset M_1^\kappa \supset M_2^\kappa \supset \dots$.

Let us note that validity of d) for the Weyl modules together with finiteness statement of part b) proves d) for any generalized Weyl module. Hence it suffices to prove d) for V_λ^κ . Recall the formula for a Sugawara operator $L_0 = \sum_{j>0} \sum_p (t^{-j}c_p)(t^j c_p) + K$, where $K = \sum_p c_p c_p$ is a standard Casimir operator of \mathfrak{g} with respect to bilinear form $-2(\kappa - \kappa_{crit})$. Since $t\mathfrak{g}[t]$ acts by 0 on any $v \in V_\lambda \subset V_\lambda^\kappa$ we get $L_0(v) = Kv = -1/2(\kappa - \kappa_{crit})^{-1}(\lambda, \lambda + 2\rho)v$. Let us denote $-1/2(\kappa - \kappa_{crit})^{-1}(\lambda, \lambda + 2\rho) =: p_\kappa(\lambda)$. Finally, using the equality $[L_0, t^k c] = kt^k c$ we see that $L_0((t^{-a_1}g_1) \dots (t^{-a_k}g_k)v) = (p_\kappa(\lambda) - (a_1 + \dots + a_k))v$. Since the set $\{(t^{-a_1}g_1) \dots (t^{-a_k}g_k)v \mid \forall i : a_i > 0, g_i \in \mathfrak{g}, v \in V_\lambda\}$ spans V_λ^κ we get the result (in this case we get a decomposition into the direct sum of eigenspaces, while for a generalized Weyl module it's essential to consider generalized eigenspaces).

Remark 5. *One corollary is that all the eigenvalues of L_0 acting on V_λ^κ start with $p_\kappa(\lambda)$ and go down discretely. Moreover, this highest component ${}_{p_\kappa(\lambda)}V_\lambda^\kappa \cong V_\lambda$ as \mathfrak{g} -module.*

Now we are ready to prove the first part of c). It suffices to show that any proper $\widehat{\mathfrak{g}}_\kappa$ -submodule $W \subset V_\lambda^\kappa$ is contained in $\bigoplus_{n \in \mathbb{N}} {}_{p_\kappa(\lambda) - n}V_\lambda^\kappa$ (it would imply the uniqueness of the maximal proper submodule and hence uniqueness of irreducible quotient). However, let us note that W is L_0 -invariant and hence $W = \bigoplus_{\chi} (W \cap {}_{\chi}V_\lambda^\kappa)$. Since \mathfrak{g} -module ${}_{p_\kappa(\lambda)}V_\lambda^\kappa \cong V_\lambda$ is simple, $W \cap {}_{p_\kappa(\lambda)}V_\lambda^\kappa$ is either 0 or the whole ${}_{p_\kappa(\lambda)}V_\lambda^\kappa$. In the latter case $W = V_\lambda^\kappa$ as V_λ generates the whole V_λ^κ under the $\widehat{\mathfrak{g}}_\kappa$ -action. This contradicts our assumption on W to be proper and the statement follows.

Let us now prove $L_\lambda^\kappa(1) = V_\lambda$, which in particular immediately implies $L_\lambda^\kappa \not\cong L_\mu^\kappa$ for $\lambda \neq \mu$. Inclusion $L_\lambda^\kappa(1) \supset V_\lambda$ is obvious. Assume that $V_\lambda \neq L_\lambda^\kappa(1)$. Since the action of \mathfrak{g} on $L_\lambda^\kappa(1)$ integrates to an action of group G there exists $\mu \in \Lambda^+$ and a monomorphism of G -modules $\gamma : V_\mu \rightarrow L_\lambda^\kappa(1)$, s.t. $\text{Im}(\gamma) \cap V_\lambda = 0$. Frobenius reciprocity provides a $\widehat{\mathfrak{g}}_\kappa$ -homomorphism $\Upsilon : V_\mu^\kappa \rightarrow L_\lambda^\kappa$, whose restriction to $V_\mu \subset V_\lambda^\kappa$ coincides with γ . Since L_λ^κ is irreducible, Υ nonzero and V_μ^κ has a unique irreducible quotient we get that $L_\mu^\kappa = L_\lambda^\kappa$ and Υ is the natural homomorphism $V_\mu^\kappa \rightarrow L_\mu^\kappa \cong L_\lambda^\kappa$. Since $\text{Im}(\gamma) \cap V_\lambda = 0$ we easily get $\text{Im}(\Upsilon) \cap V_\lambda = 0$. Recalling the eigenvalues appearing in a decomposition of part d) in the case of a Weyl module, we see that $p_\kappa(\lambda) - p_\kappa(\mu) \in \mathbb{N}$, $p_\kappa(\mu) - p_\kappa(\lambda) \in \mathbb{Z}_{\geq 0}$ providing a contradiction.

Finally, let's handle b). Because of a) it suffices to show that any Weyl module V_λ^κ has finite length. Let us introduce the set $S := \{\delta \in \Lambda^+ \mid p_\kappa(\lambda) - p_\kappa(\delta) \in \mathbb{N}\}$. This set is finite.³ Since all generalized eigenspaces of V_λ^κ are finite dimensional it suffices to check that for any $\widehat{\mathfrak{g}}_\kappa$ -submodules $M_2 \subsetneq M_1 \subsetneq V_\lambda^\kappa$ there exists $\delta \in S$, s.t. $(M_1/M_2) \cap {}_{p_\kappa(\delta)}V_\lambda^\kappa \neq 0$. Indeed, this would immediately imply that the length of V_λ^κ is bounded from above by $\sum_{\delta \in S} \dim({}_{p_\kappa(\delta)}V_\lambda^\kappa)$. As $M_1/M_2 \neq 0$ and the set of generalized eigenvalues of L_0 acting on $\widetilde{M} := M_1/M_2$ is discrete and bounded from above (since it is so for the whole V_λ^κ) we can pick $\chi \in \Lambda$, s.t. ${}_{\chi}\widetilde{M} \neq 0$ and $\forall \xi : {}_{\xi}\widetilde{M} \neq 0$ inequality $\Re(\chi) \geq \Re(\xi)$ holds (we use notation $\Re(t)$ for the real part of

³Presenting κ in the form $\kappa = c \cdot \kappa_{Kill}$ there are two possibilities: $c+1/2 \notin \mathbb{Q}$, and so $S = \{\lambda\}$, or $c+1/2 \in \mathbb{Q}_{<0}$ implying finiteness of S because of the positivity of bilinear form $-1/2(\kappa - \kappa_{crit})^{-1}$ and discreteness of Λ .

$t \in \mathbb{C}$). Then from the commutation relation $[L_0, tc] = tc$, $c \in \mathfrak{g}$, we see that $t\mathfrak{g}$ should act by zero on ${}_{\chi}\widetilde{M}$, because of the maximality of χ . Hence ${}_{\chi}\widetilde{M} \subset \widetilde{M}(1)$. Pick a monomorphism of G -modules $\phi : V_{\nu} \rightarrow {}_{\chi}\widetilde{M} \subset \widetilde{M}(1)$ for some $\nu \in \Lambda^+$. By the Frobenius reciprocity we get a $\widetilde{\mathfrak{g}}_{\kappa}$ -homomorphism $\Phi : V_{\nu}^{\kappa} \rightarrow \widetilde{M}$. Then $p_{\kappa}(\lambda) - p_{\kappa}(\nu) \in \mathbb{N}$ (it can't be 0 since $M_1 \neq V_{\lambda}^{\kappa}$). So $\nu \in S$.

Remark 6. *Plugging in the above proof $M_2 = 0$ we get that if $\forall \mu \in \Lambda^+ : p_{\kappa}(\lambda) - p_{\kappa}(\mu) \notin \mathbb{N}$ then V_{λ}^{κ} is irreducible. In particular, V_0^{κ} is irreducible.*

□

3. COINVARIANTS

One of the key constructions in this seminar is a *fusion product*. Let us recall it.

Definition 6. a) For a finite subset S of \mathbb{P}^1 we denote by A_S the ring of rational functions on \mathbb{P}^1 regular outside S and write $\mathfrak{g}_{out,S} := \mathfrak{g} \otimes_{\mathbb{C}} A_S$,

b) A coordinate system at S is a family $\Psi = \{\phi_s\}$, $\phi_s : \widehat{\mathbb{P}}_s^1 \rightarrow \widehat{D}$ of isomorphisms between the completions of $\widehat{\mathbb{P}}_s^1$ of \mathbb{P}^1 at $s \in S$ with a formal disc \widehat{D} ,

c) We denote by $\mathfrak{g}_{\kappa,S}$ the central extension $0 \rightarrow \mathbb{C} \rightarrow \mathfrak{g}_{\kappa,S} \rightarrow \bigoplus_{s \in S} \mathfrak{g}((t)) \rightarrow 0$, which is the quotient of $\bigoplus_{s \in S} \widehat{\mathfrak{g}}_{\kappa}$ by the kernel of the addition map $\bigoplus_{s \in S} \mathbb{C} \rightarrow \mathbb{C}$.

Remark 7. A coordinate system Ψ defines an embedding $\beta_{\Psi} : \mathfrak{g}_{out,S} \rightarrow \bigoplus_{s \in S} \mathfrak{g}((t))$. As an immediate consequence of the *sum of residues formula* it lifts uniquely to a Lie algebra homomorphism $\widetilde{\beta}_{\Psi} : \mathfrak{g}_{out,S} \rightarrow \mathfrak{g}_{\kappa,S}$.

d) Given a family of $\widehat{\mathfrak{g}}_{\kappa}$ -modules $\{(\rho_s, V_s)\}_{s \in S}$ and a coordinate system Ψ at S we denote by $C(\{\rho_s\}, \Psi)$ the space of coinvariants of $\bigotimes_{s \in S} V_s$ by the action of the Lie algebra $\widetilde{\beta}_{\Psi}(\mathfrak{g}_{out,S})$.⁴

We have the following trivial Proposition:

Proposition 2. a) For any family of finite dimensional $G[[t]]$ -modules $\{N_s\}_{s \in S}$ and a coordinate system Ψ at S the space $C(\{N_s^{\kappa}\}, \Psi)$ is isomorphic to the space of coinvariants of $\bigotimes_{s \in S} N_s$ under the diagonal action of \mathfrak{g} ,

b) For any family of $\widehat{\mathfrak{g}}_{\kappa}$ -representations $\{(\rho_s, V_s)\}$ which are quotients of the generalized Weyl modules and a coordinate system Ψ at S , the space of coinvariants $C(\{\rho_s\}, \Psi)$ is finite dimensional.

Proof. a) Can be easily proved by induction, using the fact that given any principal parts f_s at the neighborhoods of $s \in S$ there exists a meromorphic function f on \mathbb{P}^1 with poles only at S , whose principal parts at these points are given by f_s , and it is unique up to adding a constant;

b) Follows from part a) . □

Remark 8. We will see in the next section that the requirement for a $\widehat{\mathfrak{g}}_{\kappa}$ -module M to be a quotient of a generalized Weyl module is equivalent to belong to category \mathcal{O}_{κ} .

⁴Note that an action of $\bigoplus_{s \in S} \widehat{\mathfrak{g}}_{\kappa}$ on $\bigotimes_{s \in S} V_s$ factors through $\mathfrak{g}_{\kappa,S}$ and hence we get a $\mathfrak{g}_{out,S}$ -action on this module.

4. CATEGORY \mathcal{O}_κ

Now we define the basic object of this talk:

Definition 7. *The full subcategory of KL_κ consisting of finitely generated $\widehat{\mathfrak{g}}_\kappa$ -representations is called the category \mathcal{O}_κ .*

Next proposition provides us with another equivalent definitions of category \mathcal{O}_κ . In particular, we show it's equivalent to the one from [3].

Proposition 3. *The following properties of $V \in KL_\kappa$ are equivalent:*

- a) V admits a finite filtration with subquotients of the form L_ν^c for various $\nu \in \Lambda^+$,
- b) V is a quotient of a generalized Weyl module,
- c) For some N the subspace $V(N)$ is finite dimensional and $\widehat{\mathfrak{g}}_\kappa$ -generates V ,
- d) $V \in \mathcal{O}_\kappa$.

The following result is more nontrivial and supplies an intrinsic definition of the category \mathcal{O}_κ , which is generally easier to check.

Theorem 1. *$V \in KL_\kappa$ is finitely generated iff $\dim V(1) < \infty$.*

Corollary 1. *As the first application of Theorem 1 we immediately get that \mathcal{O}_κ is an abelian subcategory of KL_κ , stable under subquotients.*

Before moving to the proofs of these statements let us introduce a duality functor.

5. DUALITY FUNCTOR D

One of the key features of category \mathcal{O}_κ is existence of an anti-involution functor $D : \mathcal{O}_\kappa^{op} \rightarrow \mathcal{O}_\kappa$. In fact, we define this functor on a bigger category \mathcal{C} , where it is obviously an anti-involution, and then show that it preserves subcategory \mathcal{O}_κ .

Remark 9. *Since $KL_\kappa \subset \widehat{\mathfrak{g}}_\kappa$ -mod the action of L_0 on any $V \in KL_\kappa$ is well defined.⁵ We claim that in fact it decomposes V into the direct sum of generalized L_0 -eigenspaces $V = \bigoplus_{\chi \in \mathbb{C}} \chi V$ (which are no longer finite dimensional). To prove this it suffices to show that for any $v \in V$ there is a non-zero polynomial $P \in \mathbb{C}[t]$, s.t. $P(L_0)v = 0$. Existence of such P is guaranteed once we can find a finite dimensional vector space $W \subset V$ containing all $L_0^k(v)$, $k \geq 0$. One can easily produce such W using smoothness and integrability of V .*

Definition 8. *Define \mathcal{C} as a full subcategory of KL_κ , consisting of those modules V , s.t. in the decomposition $V = \bigoplus_{\chi \in \mathbb{C}} \chi V$ into the generalized eigenspaces of L_0 :*

- i) $\dim \chi V < \infty \forall \chi \in \mathbb{C}$,
- ii) *The set $S = \{\chi \in \mathbb{C} : \chi V \neq 0\}$ is bounded from above, i.e. there is a finite set χ_1, \dots, χ_k , s.t. $S \subset \bigcup_{1 \leq j \leq k} (\chi_j - \mathbb{Z}_{\geq 0})$.*

Note that category \mathcal{C} contains all generalized Weyl modules (see the proof of Proposition 1d) and hence any quotient of those, which are exactly objects of category \mathcal{O}_κ (see Proposition 3c).

We define functor $D : \mathcal{C}^{op} \rightarrow \mathcal{C}$ as follows. As a vector space $D(V) := \bigoplus_{\chi \in \mathbb{C}} (\chi V)^* \subset V^*$. Now our task is to define $\widehat{\mathfrak{g}}_\kappa$ -action on this vector space, s.t. $\chi(D(V)) = (\chi V)^*$. We have a natural action of $\widehat{\mathfrak{g}}_\kappa$ on the dual module V^* . However, it doesn't preserve subspace $D(V)$. In order to fix it, let us introduce an anti-involution \sharp of $\widehat{\mathfrak{g}}_\kappa$ by $\sharp : 1 \mapsto 1, t^n c \mapsto t^{-n} c$. While $\widetilde{\mathfrak{g}}_\kappa$ acts

⁵We recall that in the case of a reductive \mathfrak{g} operator $L_0 = \sum_i L_0^{(i)}$.

naturally on V^* preserving $D(V)$, we also can compose it with \sharp . The point of considering this twisted action of $\widetilde{\mathfrak{g}}_\kappa$ is that it extends by continuity to an action of $\widehat{\mathfrak{g}}_\kappa$.⁶ This defines the desired functor D . Since all ${}_\chi V$ are finite dimensional we also have $D(D(V)) = V$ for any $V \in \mathcal{C}$.

Now we want to show that D preserves a full subcategory $\mathcal{O}_\kappa \subset \mathcal{C}$. Because of the definition of \mathcal{O}_κ provided by Proposition 3a it suffices to show that $D(L_\lambda^\kappa) \in \mathcal{O}_\kappa$ for any $\lambda \in \Lambda^+$. This follows from the following lemma:

Lemma 1. *For any $\lambda \in \Lambda^+$ we have $D(L_\lambda^\kappa) = L_{\bar{\lambda}}^\kappa$, where $\bar{\lambda}$ is determined by $(V_\lambda)^* \cong_{G\text{-mod}} V_{\bar{\lambda}}$.*

Proof. Since $D(L_\lambda^\kappa) \in \mathcal{C}$ and $D \circ D = Id$ we see that $D(L_\lambda^\kappa)$ is irreducible. Moreover, generalized eigenvalues of $D(L_\lambda^\kappa)$ are bounded from above by $p_\kappa(\lambda)$. Similarly to the argument used in the proof of Proposition 1b we construct a nonzero map $\Psi : V_\nu^\kappa \rightarrow D(L_\lambda^\kappa)$ for some $\nu \in \Lambda^+$. Since $D(L_\lambda^\kappa)$ is irreducible, Ψ nonzero and V_ν^κ admits a unique simple quotient, we see that $D(L_\lambda^\kappa) \cong L_\nu^\kappa$. Moreover, restricting to the action of \mathfrak{g} -modules we get an isomorphism of \mathfrak{g} -modules $V_\nu \cong_{p_\kappa(\lambda)} (V_\nu^\kappa) \cong_{p_\kappa(\lambda)} (D(L_\lambda^\kappa)) \cong V_{\bar{\lambda}}$ implying $D(L_\lambda^\kappa) \cong L_{\bar{\lambda}}^\kappa$. \square

As already mentioned above this implies

Corollary 2. *Anti-involution D preserves subcategory \mathcal{O}_κ .*

Now we will show that when restricted to category $\mathcal{O}_\kappa \subset \mathcal{C}$ duality functor D can be written in another way, as it was presented in Kazhdan's talk and in [3].

Lemma 2. *For any $V \in \mathcal{O}_\kappa$ we have $D(V) \cong (((V^*)|_{\widehat{\mathfrak{g}}_\kappa})^\sharp)_{sm}^{int}$.*

We denote the right hand side of this expression by $D'(V)$. Let us explain why it is a natural object, besides its annoying definition. We wish to consider $(V^*)^\sharp$ as a $\widehat{\mathfrak{g}}_\kappa$ -module, but by the reasons explained in the definition of $D(V)$ it can be viewed only as a $\widetilde{\mathfrak{g}}_\kappa$ -module. However, after taking the smooth part the corresponding $\widetilde{\mathfrak{g}}_\kappa$ -submodule integrates to an action of $\widehat{\mathfrak{g}}_\kappa$.

In the proof of this lemma, we will need the following result, which characterizes $D'(V)$ (and hence, at the end of the day, $D(V)$) in an abstract way without this ugly formula. Moreover, it is a starting point for Giorgia's talk:

Proposition 4. *Let $S = \{0, \infty\} \subset \mathbb{P}^1$ and a coordinate system $\Psi = (\phi_0, \phi_\infty)$ be determined by choosing coordinate $\frac{z_1}{z_2}$ at 0 and $\frac{z_2}{z_1}$ at ∞ (here $(z_1 : z_2)$ are the standard coordinates on \mathbb{P}^1). Then for any $V, V' \in KL_\kappa$ we have $\text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V', D'(V)) = C(\{V', V\}, \Psi)^*$ as vector spaces.*

Remark 10. *In other words, $D'(V)$ is an object of category KL_κ representing the functor $C(\{-, V\}, \Psi)^*$ from KL_κ to $\text{Vect}_{\mathbb{C}}$.*

Proof. Since $V' \in KL_\kappa$ any homomorphism $V' \rightarrow (V^*)^\sharp = (V^*|_{\widehat{\mathfrak{g}}_\kappa})^\sharp$ automatically factors through $D'(V)$. Hence, $\text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V', D'(V)) = \text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V', (V^*|_{\widehat{\mathfrak{g}}_\kappa})^\sharp)$. The basic linear algebra provides an isomorphism $\text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V', (V^*|_{\widehat{\mathfrak{g}}_\kappa})^\sharp) \cong \text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V' \otimes V^*|_{\widehat{\mathfrak{g}}_\kappa}, \mathbb{C})$. Finally, $\text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V' \otimes V^*|_{\widehat{\mathfrak{g}}_\kappa}, \mathbb{C}) = C(\{V', V\}, \Psi)^*$, finishing the proof. \square

We finish this section with the proof of Lemma 2.

Proof. Since both $D(V)$ and $D'(V)$ when restricted to $\widetilde{\mathfrak{g}}_\kappa$ can be viewed as submodules of $((V^*)|_{\widetilde{\mathfrak{g}}_\kappa})^\sharp$ it suffices to check $D'(V) = D(V)$ as vector subspaces.

To prove $D'(V) \supset D(V)$ we need to show $({}_\lambda V)^* \subset (((V^*)|_{\widehat{\mathfrak{g}}_\kappa})^\sharp)_{sm}$ for any λ , i.e. that there exists N , s.t. $Q_N \cdot ({}_\lambda V)^* = 0$. This is equivalent to $Q_N^\sharp \cdot V$ has zero projection on ${}_\lambda V \subset V$ (where

⁶The reason being that the set $S = \{\chi : {}_\chi V \neq 0\}$ is bounded from above and $t^k \mathfrak{g} : {}_\chi V \rightarrow {}_{\chi+k} V$.

\mathcal{Q}_N^\sharp is the image of \mathcal{Q}_N under \sharp). This immediately follows from $\mathcal{O}_\kappa \subset \mathcal{C}$, any object of which has only finite number of L_0 -eigenvalues, whose real part is bigger then any fixed number.

Now let us prove " \supset ". We refer the reader to [3], 2.23, for a straightforward proof (the key observation is that for any N there exists a finite set $S \subset \{\lambda : {}_\lambda V \neq 0\}$, s.t. $\mathcal{Q}_N^\sharp \cdot V \supset \bigoplus_{\mu \notin S} {}_\mu V$).

Another approach (suggested by Dennis) is to use Proposition 4. The following claim can be proved using techniques, which will be introduced in Andrei's talk: given a $\tilde{\mathfrak{g}}_\kappa$ -homomorphism $\phi : W \otimes V \rightarrow \mathbb{C}$ (with a trivial module structure on the target and the one on the source given by $\tilde{\beta}_\Psi(\mathfrak{g}_{out,0,\infty})$) we have $\phi|_{\lambda W \otimes {}_\mu V} = 0$ unless $\lambda = \mu$. This fact implies that given any $W \in \text{KL}_\kappa$ and a $\hat{\mathfrak{g}}_\kappa$ -homomorphism $\psi : W \rightarrow D'(V) \subset V^*$ we should have $\psi({}_\lambda W) \subset {}_\lambda V^*$ and hence $\psi : W \rightarrow V^*$ factors through $\bigoplus_\lambda V^*$ implying $D'(V) \subset D(V)$. \square

6. PROOF OF PROPOSITION 3

We start from the equivalence of b) and d). Since any generalized Weyl module is finitely generated and lives in category KL_κ the same holds for it's quotients, proving $b) \Rightarrow d)$. Vice-versa, given a module $V \in \mathcal{O}_\kappa$ we can find a finite set $S \in V$, which generates V under $\hat{\mathfrak{g}}_\kappa$ -action. Since the action of \mathfrak{g} integrates to G and $t\mathfrak{g}[[t]]$ acts locally nilpotently there exists a finite dimensional $G[[t]]$ -submodule M of V containing S . The natural homomorphism $M^\kappa \rightarrow V$ is, in fact, an epimorphism as M generates V . This implies $d) \Rightarrow b)$.

Let us show $c) \Rightarrow b)$ now. It is easy to see that $V(N)$ is a $G[[t]]$ -module as $(t^{a_1} g_1) \dots (t^{a_N} g_N)$ acts trivially on $V(N)$ for any $a_i \in \mathbb{N}, g_i \in \mathfrak{g}$. This provides a homomorphism of $\hat{\mathfrak{g}}_\kappa$ -modules $\Upsilon : (V(N))^\kappa \rightarrow V$. Since $V(N)$ generates V , homomorphism Υ is surjective.

In turn, implication $b) \Rightarrow a)$ is obvious. Indeed, any generalized Weyl module is of finite length by Proposition 1b). Moreover, its subsequent subquotients are irreducibles, whose set of generalized eigenvalues is bounded from above. Then the argument used in the proof of Lemma 1 shows that this irreducible module must be of the form L_ν^κ for some $\nu \in \Lambda^+$.

Finally, let us prove $a) \Rightarrow c)$.

Possessing a finite filtration with quotients L_ν^κ , which are finitely generated, V is finitely generated as well. Since $V = \bigcup V(N)$ we can find a particular N , s.t. $V(N)$ contains all these generators. As $V(N)$ generates V we only need to check $\dim V(N) < \infty$. This follows from

Lemma 3. *For any module V satisfying conditions of Proposition 3a), the subspace $V(N)$ is finite dimensional.*

Proof. Since for any N we have an exact sequence $0 \rightarrow V(1) \rightarrow V(N) \rightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{g}, V(N-1))$ of \mathbb{C} -vector spaces with the last map defined by $x \mapsto (c \mapsto (tc)x)$ and \mathfrak{g} is finite dimensional it's enough to prove that $\dim V(1) < \infty$.

Note that given a short exact sequence $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ in the category KL_κ we have an exact sequence of \mathbb{C} -vector spaces: $0 \rightarrow V_1(1) \rightarrow V(1) \rightarrow V_2(1)$, reducing to the case $V = L_\lambda^\kappa$. However, $L_\lambda^\kappa(1) = V_\lambda$ by Proposition 1c) and hence is finite dimensional.

7. PROOF OF THEOREM 1

In this section we switch between equivalent definitions of the category \mathcal{O}_κ from Proposition 3. In particular, we use criteria that $V \in \mathcal{O}_\kappa$ iff it satisfies conditions of part a).

One direction in Theorem 1 is easy. Namely, if $V \in \mathcal{O}_\kappa$ then $\dim V(1) < \infty$ by Lemma 3. The other direction is more involved.

In the rest of the proof we are assuming that the level of $\kappa - \kappa_{crit}$ is rational and hence negative, since the irrational case is easy to handle.

First we introduce the following filtration \mathcal{O}_κ^k on \mathcal{O}_κ .

Definition 9. For a positive $k \in \mathbb{R}_{>0}$ we define

- (a) A set $F^k = \{\lambda \in \Lambda^+ : p_\kappa(\lambda) < k\}$ (this set is finite as $\kappa - \kappa_{crit}$ is negative),
- (b) A partial order \preceq on F^k by saying $\mu \preceq \nu$ if either $\mu = \nu$ or $p_\kappa(\mu) < p_\kappa(\nu)$,
- (c) The full subcategory $\mathcal{O}_\kappa^k \subset \mathcal{O}_\kappa$, whose objects are exactly those modules V from category \mathcal{O}_κ with all composition factors being L_λ^κ , $\lambda \in F^k$.

Step 1: Category \mathcal{O}_κ^k is closed under extensions in the category \mathcal{O}_κ ;

Step 2: If $V \in \mathcal{O}_\kappa^k$ then $D(V) \in \mathcal{O}_\kappa^k$ (use $D(L_\lambda^\kappa) = L_{\bar{\lambda}}^\kappa$ and $p_\kappa(\lambda) = p_\kappa(\bar{\lambda})$);

Step 3: Argument from the proof of Proposition 1b proves that if $\lambda \in F^k$ then $V_\lambda^\kappa \in \mathcal{O}_\kappa^k$;

Step 4: Assuming λ is a maximal element of F^k for the order \preceq , the canonical map $\pi_\lambda : V_\lambda^\kappa \rightarrow L_\lambda^\kappa$ is a projective cover of L_λ^κ in the category \mathcal{O}_κ^k .

Remark 11. We remind that the homomorphism $\pi : P \rightarrow X$ of A -modules is called a projective cover of X if P is projective, π -epimorphism and for any A -submodule $P_0 \subset P$ the restricted map $\pi| : P_0 \rightarrow X$ is no longer an epimorphism.

Proof. First we prove that the restriction map $\text{Hom}_{\widehat{\mathfrak{g}}_\kappa\text{-mod}}(V_\lambda^\kappa, X) \rightarrow \text{Hom}_{G\text{-mod}}(V_\lambda, p_\kappa(\lambda)X)$ is in fact a bijection for any $X \in \mathcal{O}_\kappa^k$. For this it suffices to prove $p_\kappa(\lambda)X \subset X(1)$. If not, $p_\kappa(\lambda)+1X \neq 0$ contradicting $X \in \mathcal{O}_\kappa^k$ together with maximality of λ .

Now we easily derive the statement of Step 4. It is clear that any proper submodule $V' \subset V_\lambda^\kappa$ doesn't map surjectively onto L_λ^κ under the restriction of π_λ . So we only need to check that V_λ^κ is projective in category \mathcal{O}_κ^k , i.e. given any surjective homomorphism $\rho : Y \rightarrow X$ and a homomorphism $f : V_\lambda^\kappa \rightarrow X$ in category \mathcal{O}_κ^k we need to provide a homomorphism $f' : V_\lambda^\kappa \rightarrow Y$, s.t. $\rho f' = f$. However, since in the category of finite dimensional G -modules all objects are projective and restriction $\rho^r : p_\kappa(\lambda)Y \rightarrow p_\kappa(\lambda)X$ is surjective there exists a G -homomorphism $h : V_\lambda \rightarrow p_\kappa(\lambda)Y$ s.t. $\rho^r h = f^r$. Using the bijection established above this provides a $\widehat{\mathfrak{g}}_\kappa$ -homomorphism $f' : V_\lambda^\kappa \rightarrow Y$, whose restriction to V_λ coincides with h . Since restrictions of $\rho f'$, $f : V_\lambda^\kappa \rightarrow X$ to V_λ are equal, the bijection statement applied once again implies $\rho f' = f$. \square

Step 5: For any $\lambda \in \Lambda^+$ we have $\text{Hom}(L_\lambda^\kappa, L_\lambda^\kappa) = \mathbb{C}$ (Schur lemma).

Now a **Key Step** of the proof comes:

Step 6: We have $\text{Ext}_{\text{KL}_\kappa}^{>0}(V_\lambda^\kappa, D(V_\mu^\kappa)) = 0$ for any Weyl modules $V_\lambda^\kappa, V_\mu^\kappa$ ($\lambda, \mu \in F^k$).

Proof. Since $(\text{Ind}_{G[[t]]}^{\text{KL}_\kappa}, \text{Res}_{\text{KL}_\kappa}^{G[[t]]})$ is a pair of adjoint functors between categories KL_κ and $G[[t]]\text{-mod}$ and $\text{Ind}_{G[[t]]}^{\text{KL}_\kappa}$ is an exact functor the general result from homological algebra implies $\text{Ext}_{\text{KL}_\kappa}^i(\text{Ind}(M), W) = \text{Ext}_{G[[t]]\text{-mod}}^i(M, W)$.

In particular we get $\text{Ext}_{\text{KL}_\kappa}^i(V_\lambda^\kappa, D(V_\mu^\kappa)) = \text{Ext}_{G[[t]]\text{-mod}}^i(V_\lambda, D(V_\mu^\kappa)|_{G[[t]]})$.

Now we claim that $D(V_\mu^\kappa)|_{G[[t]]} = \text{Coind}_G^{G[[t]]}(V_{\bar{\mu}})$.

Remark 12. Recall that co-induction is a right adjoint functor to restriction. In the case of subalgebra $B \subset A$ it is defined by $\text{Coind}_B^A(M) = \text{Hom}_B(A, M)$ for any B -module M .

In order to check $D(V_\mu^\kappa)|_{G[[t]]} = \text{Coind}_G^{G[[t]]}(V_{\bar{\mu}})$ it suffices to show

$$\text{Hom}_{G[[t]]\text{-mod}}(M, D(V_\mu^\kappa)|_{G[[t]]}) = \text{Hom}_{G\text{-mod}}(M, V_{\bar{\mu}})$$

for any $G[[t]]$ -module M . This follows from the following sequence of equalities:

$$\begin{aligned} \text{Hom}_{G[[t]]\text{-mod}}(M, D(V_\mu^\kappa)|_{G[[t]]}) &= \text{Hom}_{\text{KL}_\kappa}(\text{Ind}_{G[[t]]}^{\text{KL}_\kappa}(M), D(V_\mu^\kappa)) = C(\{\text{Ind}_{G[[t]]}^{\text{KL}_\kappa}(M), V_\mu^\kappa\}, \Psi)^* \\ &= ((M \otimes V_\mu)_\mathfrak{g})^* = \text{Hom}_{\mathfrak{g}\text{-mod}}(M, V_{\bar{\mu}}) = \text{Hom}_{G\text{-mod}}(M, V_{\bar{\mu}}), \end{aligned}$$

where we have used both Propositions 2 and 4, and integrability of \mathfrak{g} -action to G .

Since the pair $(Res_{G[[t]]}^G, Coind_G^{G[[t]])}$ is a pair of adjoint functor between categories $G[[t]]\text{-mod}$ and $G\text{-mod}$ and $Coind_G^{G[[t]]}$ is an exact functor we have $Ext_{G[[t]]\text{-mod}}^i(V_\lambda, Coind_G^{G[[t]]}(V_{\bar{\mu}})) = Ext_{G\text{-mod}}^i(V_\lambda, V_{\bar{\mu}})$. As there are no higher Ext -groups between finite dimensional representations of G we get the result. \square

Remark 13. *In fact, it suffices to know this vanishing condition for $Ext^{1,2}$.*

Now general yoga implies the following result (see [4]):

Claim 1. *For any $\lambda \in F^k$ there is a projective object $P_\lambda^\kappa \in \mathcal{O}_\kappa^k$, s.t.*

a) *We have $\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, L_\mu^\kappa) = \delta_{\lambda, \mu}$ for any $\lambda, \mu \in F^k$;*

b) *Module P_λ^κ admits a finite filtration with quotients of the form V_μ^κ ($\mu \in F^k$) (such filtration is called standard) and the number of occurrences of V_μ^κ equals to $[V_\mu^\kappa : L_\lambda^\kappa]$ (where for $V \in \mathcal{O}_\kappa$ the notation $[V : L_\lambda^\kappa]$ stands for the number of subquotients in a composition series of V isomorphic to L_λ^κ).*

We will present a sketch of the proof at the end of this section, but let us first deduce the proof of Theorem 1. As an immediate corollary of the above claim we get:

Corollary 3. *For any $V \in \mathcal{O}_\kappa$ and any $\lambda \in \Lambda^+$ the following inequality holds: $[V : L_\lambda^\kappa] \leq \sum_\mu [V_\mu^\kappa : L_\lambda^\kappa][V(1) : V_\mu]$ (where $[V(1) : V_\mu]$ equals to $\dim \text{Hom}_{\mathfrak{g}}(V_\mu, V(1))$, or equivalently, the multiplicity of V_μ in the composition series of $V(1)$ as a \mathfrak{g} -module).*

Proof. First we choose k , s.t. $V \in \mathcal{O}_\kappa^k$. Since module P_λ^κ is projective in \mathcal{O}_κ^k the function $\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, -)$ is additive in \mathcal{O}_κ^k . Hence

$$\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, V) = \sum_\mu \dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, L_\mu^\kappa)[V : L_\mu^\kappa] = \sum_\mu \delta_{\lambda, \mu}[V : L_\mu^\kappa] = [V : L_\lambda^\kappa].$$

Also since for any short exact sequence of modules $0 \rightarrow X_1 \rightarrow X \rightarrow X_2 \rightarrow 0$ in \mathcal{O}_κ^k we have $\dim \text{Hom}_{\mathcal{O}_\kappa}(X, V) \leq \dim \text{Hom}_{\mathcal{O}_\kappa}(X_1, V) + \dim \text{Hom}_{\mathcal{O}_\kappa}(X_2, V)$ the following inequality holds:

$$\dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, V) \leq \sum_\mu [P_\lambda^\kappa : V_\mu^\kappa] \dim \text{Hom}_{\mathcal{O}_\kappa}(V_\mu^\kappa, V) = \sum_\mu [V_\mu^\kappa : L_\lambda^\kappa] \dim \text{Hom}_{\mathfrak{g}\text{-mod}}(V_\mu, V(1))$$

finishing the proof. \square

Now we are ready to prove the nontrivial part of Theorem 1.

First we note that for any module $V \in \text{KL}_\kappa$ there exists a filtration $\{V^i\}$ of V by finitely generated submodules (i.e. $V^i \in \mathcal{O}_\kappa$), s.t. $V = \bigcup_n V^n$. Hence to show that $V \in \mathcal{O}_\kappa$ it suffices to bound the length of any $\widehat{\mathfrak{g}}_\kappa$ -submodule W of V by some universal constant. This can be easily performed applying previous Corollary. Namely for any submodule W its length $l(W)$ is bounded by

$$l(W) \leq \sum_{\lambda, \mu} [V_\mu^\kappa : L_\lambda^\kappa][W(1) : V_\mu] \leq \sum_{\lambda, \mu} [V_\mu^\kappa : L_\lambda^\kappa][V(1) : V_\mu].$$

Hence if $\dim V(1) < \infty$ only finite number of μ give a nonzero input in $[V(1) : V_\mu]$ and for each of them only finite number of λ satisfy $[V_\mu^\kappa : L_\lambda^\kappa] \neq 0$. This bounds $l(W)$ by a universal constant as desired. \square

Sketch of the proof of Claim 1 (see [1], section 4)

a) As turns out, it is easier to work with injective objects. So we will prove that for any $\lambda \in F^k$ there exists an injective object $I_\lambda^\kappa \in \mathcal{O}_\kappa^k$ s.t. $\dim \text{Hom}_{\mathcal{O}_\kappa}(L_\mu^\kappa, I_\lambda^\kappa) = \delta_{\lambda, \bar{\mu}}$ for any $\lambda, \mu \in F^k$ and I_λ^κ admits a *co-standard filtration* with quotients of the form $D(V_\mu^\kappa)$ ($\mu \in F^k$).

Since KL_κ is closed under filtered colimits, L_λ^κ admits an injective hull I_λ^κ in category KL_κ . Our goal is to show that I_λ^κ admits a co-standard filtration. This can be proved similarly to Proposition 4.13 [1] using an intrinsic characterization of modules admitting co-standard filtrations:

Lemma 4. *An object $M \in \mathcal{O}_\kappa^k$ admits a co-standard filtration iff $\text{Ext}_{\text{KL}_\kappa}^{1,2}(V_\mu^\kappa, M) = 0 \forall \mu \in F^k$.*

Finally we set $P_\lambda^\kappa := D(I_\lambda^\kappa)$. Since D is an anti-automorphism of \mathcal{O}_κ^k and $D(L_\lambda^\kappa) = L_\lambda^\kappa$ part a) and existence of a standard filtration on P_λ^κ follow. It's finiteness automatically follows from the formula for multiplicities, obtained below.

b) In what follows we denote the number of these occurrences in subject by $[P_\lambda^\kappa : V_\mu^\kappa]$. Then we have the following sequence of equalities:

$$[P_\lambda^\kappa : V_\mu^\kappa] \stackrel{(1)}{=} \dim \text{Hom}_{\mathcal{O}_\kappa}(P_\lambda^\kappa, D(V_\mu^\kappa)) \stackrel{(2)}{=} [D(V_\mu^\kappa) : L_\lambda^\kappa] \stackrel{(3)}{=} [V_\mu^\kappa : L_\lambda^\kappa].$$

Here (1) follows from the vanishing of higher *Ext*-groups from Step 6 and an equality

$$\dim \text{Hom}(V_\nu^\kappa, D(V_\mu^\kappa)) = \dim C(\{V_\nu^\kappa, V_\mu^\kappa\}, \Psi) = \dim((V_\nu \otimes V_\mu)/\mathfrak{g}) = \delta_{\nu, \mu}$$

where we have used both Propositions 2, 4.

Equality (2) follows obviously from $\dim \text{Hom}(P_\lambda^\kappa, L_\mu^\kappa) = \delta_{\lambda, \mu}$ and $\dim \text{Hom}(P_\lambda^\kappa, -)$ being an additive function.

Finally (3) follows from $D(L_\lambda^\kappa) = L_\lambda^\kappa$ and an equality $[V_\lambda^\kappa : L_\mu^\kappa] = [V_\lambda^\kappa : L_\mu^\kappa]$. ■

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